

ON LOCATION OF ZEROES OF THE FIRST DERIVATIVE OF A POLYNOMIAL

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ABSTRACT. Let $p(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_1z + a_0$ be a complex polynomial, $\deg p = n \geq 2$, and let z_1, \dots, z_n be the zeros of $p(z)$, counting multiplicities. Let c be the centroid of the first $n-1$ zeros z_1, \dots, z_{n-1} and let C be a disc centered at c which contains z_1, \dots, z_{n-1} . Then the same disc contains at least

$$\left\lceil \frac{n-1}{2} \right\rceil$$

zeros of the polynomial $p'(z)$.

1. INTRODUCTION AND STATEMENT OF THE RESULT

One of the main problems in the theory of geometry of polynomials is the following one: given some information on the location of the zeros of a polynomial $p(z)$, deduce something on the location of one or all zeros of the polynomial $p(z)$. There are many results of this type, we mention Gauss-Lucas theorem and Grace-Heawood theorem as famous theorems in this area. In this note we prove the following result, which deals with the same problem.

Theorem 1. *Let $p(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_1z + a_0$ be a complex polynomial, $\deg p = n \geq 2$, and let z_1, \dots, z_n be the zeros of $p(z)$, counting multiplicities. Let*

$$c = \frac{z_1 + \cdots + z_{n-1}}{n-1}$$

and let C be a disc centered at c which contains z_1, \dots, z_{n-1} . Then the same disc contains at least

$$\left\lceil \frac{n-1}{2} \right\rceil$$

zeros of the polynomial $p'(z)$.

Note that the interesting cases are $n \geq 3$, for $n = 2$ there is nothing to prove.

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The proof we give uses Walsh Coincidence Theorem, see [2], for readers' convenience we state it below.

Theorem 2 (Walsh). *Let $f(x_1, \dots, x_n)$ be a complex polynomial, which is linear and symmetric in each of the variables x_1, \dots, x_n , and has total degree n . If n points x_1^0, \dots, x_n^0 belong to a disc C (which can be either open or closed), then there is a point x in C such that $f(x_1^0, \dots, x_n^0) = f(x, x, \dots, x)$.*

2. PROOF

If z_n belongs to C , then the result follows from the Gauss-Lucas theorem. Hence we assume z_n is not in C . Clearly, we can assume $z_n = 0$. Moreover, using suitable rotation, we can assume $c > 0$.

Now, let y_1, \dots, y_{n-1} be the zeros of $p'(z)$, counting multiplicities. Then we have, for every $1 \leq i \leq n-1$,

$$\sum_{k=1}^n \prod_{j \neq k} (y_i - z_j) = 0.$$

The expression above is linear and symmetric in z_1, \dots, z_{n-1} and has total degree $n-1$, hence we can apply Theorem 2. Therefore there is, for each $1 \leq i \leq n-1$, a point c_i in C such that

$$(n-1)y_i(y_i - c_i)^{n-2} + (y_i - c_i)^{n-1} = 0.$$

Note that we used here $z_n = 0$. From the above equation we conclude that, for each $1 \leq i \leq n-1$, we have $y_i = c_i$ or $y_i = c_i/n$. Assume the first possibility occurs k times, by renumbering we can assume $y_i = c_i$ for $1 \leq i \leq k$. Clearly it suffices to show that $k > (n-3)/2$. By the above we have

$$y_i = \frac{c_i}{n}, \quad k+1 \leq i \leq n-1.$$

Using Viet's rule, we have

$$\frac{1}{n} \sum_{i=1}^n z_i = \frac{1}{n-1} \sum_{i=1}^{n-1} y_i,$$

which gives

$$\frac{n-1}{n}c = \frac{1}{n-1} \left(\sum_{i=1}^k y_i + \sum_{i=k+1}^{n-1} \frac{c_i}{n} \right). \quad (1)$$

Let r be the radius of C . Then we have

$$\Re\left(\frac{c_i}{n}\right) \leq \frac{r+c}{n}, \quad \Re y_j \leq r+c, \quad 1 \leq j \leq k, \quad k+1 \leq i \leq n-1.$$

Since $c > 0$, we can take real parts in (1) and obtain

$$\frac{n-1}{n}c \leq \frac{1}{n-1} \left(\sum_{j=1}^k (r+c) + \sum_{i=k+1}^{n-1} \frac{r+c}{n} \right) = \frac{r+c}{n-1} \left(k + \frac{n-k-1}{n} \right),$$

which is equivalent to

$$(n-k-2)c \leq r(1+k).$$

Since C does not contain the origin, we have $r < c$ and this gives the needed estimate $n-k-2 < 1+k$.

REFERENCES

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